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DIMENSION THEORY OF COMMUTATIVE RINGS WITHOUT IDENTITY

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1. Introduction

Throughout this paper all rings are assumed to be commutative. It is well known that a ring R can be embedded in a ring with identity. This fact was proved by Dorroh [5], using the following construction. Let R^* be the set of ordered pairs (r, n), where $r \in R$ and $n \in \mathbf{Z}$. If (r_1, n_1), (r_2, n_2) are in R^*, we define

$$
\begin{aligned}
(r_1, n_1) &= (r_2, n_2) \quad \text{if and only if} \quad r_1 = r_2,\ n_1 = n_2\,,\\
(r_1, n_1) + (r_2, n_2) &= (r_1 + r_2, n_1 + n_2)\,,\\
(r_1, n_1)(r_2, n_2) &= (r_1 r_2 + n_2 r_1 + n_1 r_2, n_1 n_2)\,.
\end{aligned}
$$

Under these operations, R^* is a ring with identity $(0, 1)$ and the mapping $r \to (r, 0)$ is an imbedding of R into R^*. We note that, as a ring, R^* is generated by R and the identity element $(0, 1)$. Such a ring will be called a unital extension of R that is, a ring S with identity e is a *unital extension* of a subring R if $S = \{r + ne \mid r \in R, n \in \mathbf{Z}\}$ [8, p. 152]. It is known that R need not have a unique unital extension. In fact, if R has nonzero characteristic, then for each nonnegative integer n, there is a unital extension of R that has characteristic nk [8, p. 4].

We say that R has (Krull) *dimension n*, and write $\dim R = n$, provided there exists a chain $P_0 \subset P_1 \subset \ldots \subset P_n$ of $n + 1$ distinct proper prime ideals of R, but no such chain of $n + 2$ prime ideals. If R contains no proper prime ideals, then $\dim R = -1$.

In this paper we consider the following questions, where S is a unital extension of the ring R:

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(1) What is a complete set of representatives for the set of unital extensions of a ring R?

(2) How is the prime ideal structure of R related to that of S; in particular, what are the possibilities for $\dim S - \dim R$?

(3) What are the possibilities for $\dim S[X_1, ..., X_n] - \dim R[X_1, ..., X_n]$? In particular, is this difference "nicely" related to the difference $\dim S - \dim R$?

(4) For what sequences $\{n_i\}_{i=0}^{\infty}$ of integers does there exist a ring R without identity such that $\dim R = n_0$ and $\dim R[X_1, ..., X_i] = n_i$ for each $i \geq 0$?

A version of question (1) was considered by Brown and McCoy [4] and, in fact, the distinct R-isomorphism (Brown and McCoy use the terminology *strict isomorphism*) classes for the set of unital extensions of a ring R are determined in [4]. Some of the results of [4] are reviewed (in different notation and terminology) in Section 2. In Example 2.6 we show that unital extensions of a ring R that are R-isomorphically distinct need not be isomorphically distinct.

If S is a unital extension of R, then R is an ideal of S. Therefore, in studying unital extensions of a ring R, it is natural to pass to the more general setting in which R is merely assumed to be an ideal of the ring S. Under such an assumption, we give results in Section 3 that relate the ideal theory of R to that of S. In fact, such results appear throughout the paper and play an important role in our development. For example, if A is an ideal of the ring R, then in Proposition 5.8 we give a formula for the sequence $\{\dim A^{(n)}\}_{n=0}^{\infty}$ in terms of a certain set of prime ideals of R. Here $A^{(n)}$ denotes the polynomial ring $A[X_1, ..., X_n]$ in n indeterminates over A ($A^{(0)} = A$).

In Theorem 4.6 we provide a complete answer to the latter part of question (2). Further, in Proposition 4.8 we give a characterization of those rings R such that $\dim R = \dim S$ for each unital extension S of R.

Following the terminology of [1] and [2], we shall call a sequence $\{n_m\}_{m=0}^{\infty}$ of integers the *dimension sequence for the ring R* provided $\dim R^{(m)} = n_m$ for each integer $m \geq 0$. We prove in Theorems 5.7 and 5.9 that a sequence $\{n_m\}_{m=0}^{\infty}$ of non-negative integers is the dimension sequence for a commutative ring without identity if and only if it is the dimension sequence for a commutative ring with identity. The possible dimension sequences for rings with identity have been determined by Arnold and Gilmer in [1]. A key result in the proof of Theorem 5.7 is Proposition 5.6, which shows that if R is a ring of positive dimension with the property that R/P has an identity for each proper prime ideal P of R, then R and S have the same dimension sequence for each unital extension S of R. However, we illustrate in Example 5.14 that, in general, the dimension sequence for S need not be nicely related to that of R.

Finally, in Section 6, we extend to rings without identity some results, known for rings with identity, concerning chains of prime ideals in polynomial rings.

2. Unital extensions

It is well known that each ring R can be embedded in a ring S with identity. Following the terminology of [8], we say that the ring S with identity e is a unital extension of its subring R if $S = \{r + ne \mid r \in R, n \in \mathbf{Z}\}$; that is, if the subring of S generated by R and $\{e\}$ is S. In this section we determine, for a ring R, the set of unital extensions of R.

If S is a unital extension of the ring R, then R is an ideal of S. Hence our first result, Proposition 2.1, relates the structure of a ring to that of one of its ideals.

Proposition 2.1 uses the following terminology and notation. If R_1 is a subring of the ring R_2, then an ideal A_2 of R_2 is said to *lie over* the ideal A_1 of R_1 if $A_2 \cap R_1 = A_1$; if U, V and W are nonempty subsets of R_2, then $[U : V]_W$ denotes the set of elements x in W such that $xv \in U$ for each element v of V.

Proposition 2.1. *Assume that A is an ideal of the ring R and B is an ideal of the ring A.*

(1) *If there is an ideal of R lying over B in A, then B is an ideal of R.*

(2) *If B is an ideal of R and C is an ideal of R lying over B in A, then $C \subseteq \mathfrak{B} = [B : A]_R$; $\mathfrak{B}$ lies over B in A if and only if $[B : A]_A = B$.*

(3) *If $[B : A]_A = B$, then B is an ideal of R and*

$$\{C \mid C \text{ is an ideal of } R \text{ and } B \subseteq C \subseteq \mathfrak{B} = [B : A]_R\}$$

is the set of ideals of R lying over B in A. In this case, the structure of $\mathfrak{B}/B$ as an R-module is essentially the same as its structure as an R/A-module, and

$$\mathfrak{B}/B = \mathfrak{B}/(\mathfrak{B} \cap A) \simeq (\mathfrak{B} + A)/A \subseteq R/A\ .$$

Proof. (1) is clear. In (2), $AC \subseteq A \cap C = B$, and hence $C \subseteq \mathfrak{B}$. From the definition of $\mathfrak{B}$ it follows that $\mathfrak{B} \cap A = [B : A]_A$. Hence $\mathfrak{B} \cap A = B$ if and only if $[B : A]_A = B$.

Assume that $[B : A]_A = B$ and $r \in R$. Then

$$rB \cdot A = rA \cdot B \subseteq A \cdot B \subseteq B\,,$$

and hence $rB \subseteq [B : A]_A = B$. Therefore B is an ideal of R. The assertion of (3) concerning the set of ideals of R lying over B in A then follows easily from (2). Since $A\mathfrak{B} \subseteq B$, the structure of $\mathfrak{B}/B$ as an R-module is essentially the same as its structure as an R/A-module. □

The case of Proposition 2.1 in which R is a unital extension of A is of special interest.

Proposition 2.2. *Assume that R is a unital extension of the ring A. If B is an ideal of A, then B is an ideal of R. If C_β is an ideal of R lying over B in A, then C_β is principal modulo B; if $[B : A]_A = B$ and $\mathfrak{B} = [B : A]_R = B + (\alpha)$, then $\{B + (n\alpha)\}_{n=0}^{\infty}$ is the set of ideals of R lying over B in A.*

Proof. It is clear that B is an ideal of R. As in the proof of Proposition 2.1, the structure of C_β/B as an R-module is essentially the same as its structure as an R/A-module. Since R is a unital extension of A, $R/A \simeq \mathbf{Z}/(m)$ for some nonnegative integer m and the structure of C_β/B as an R-module is essentially the same as its structure as an abelian group. Because $C_\beta/B \simeq (C_\beta + A)/A \subseteq R/A$, it follows that C_β/B is a cyclic R-module; that is, $C_\beta = B + (x_\beta)$ for some $x_\beta \in R$. In particular, if $[B : A]_A = B$ and $\mathfrak{B} = B + (\alpha)$, then $\{B + (n\alpha)\}_{n=0}^{\infty}$ is the set of ideals of R between B and $\mathfrak{B}$, and hence $\{B + (n\alpha)\}_{n=0}^{\infty}$ is the set of ideals of R lying over B in A. □

In the notation of Proposition 2.2, we observe that if $R/A \simeq \mathbf{Z}/(m)$, where $m > 0$, then in the case in which $[B : A]_A = B$, the element α can be chosen to be of the form $a + ke$, where $a \in A$, k is a positive divisor of m, and e is the identity element of R. In this case there are only finitely many ideals of R lying over B in A; in fact, there are m/k such ideals.

If R is a ring, then the abelian group $R \oplus \mathbf{Z}$ admits a unique ring multiplication such that the mapping $r \to (r, 0)$ is a ring isomorphism and $(0, 1)$ is the identity element for the ring. The ring R^* obtained in this way is a unital extension of R; in the terminology of [8, p. 5], R^* is *the ring obtained from R by the canonical adjunction of an identity of characteristic* 0. R^* is the ring originally used by Dorroh [5] to prove that R can be embedded in a ring with identity. Identifying $(r, 0)$ with r and letting $e = (0, 1)$, the elements of R^* are uniquely expressible in the form $r + ke$, where $r \in R$ and $k \in \mathbf{Z}$. If S is a unital extension of R with identity element e', then the mapping $r + ne \to r + ne'$ of R^* onto S is an R-homomorphism. Hence if $\{C_\beta\}$ is the set of ideals of R^* lying over (0) in R, then $\{R^*/C_\beta\}$ is the family of unital extensions of R. It follows from Proposition 2.2 and its proof that each C_β is principal as an ideal of R^*, say $C_\beta = (r_\beta + k_\beta e)$, where $k_\beta \geqslant 0$, and the ideal generated by $r_\beta + k_\beta e$ coincides with the additive group generated by $r_\beta + k_\beta e$. If some k_β is positive, then the smallest such integer k_{β_0} is a divisor of each k_β, and is called the *mode* of R. If each k_β is zero, then we say that R has *mode* 0; this is the terminology of Brown and McCoy [4]. In words, R has mode zero if no nonzero element of R acts like a nonzero integer under multiplication, and in the contrary case, the mode of R is the smallest positive integer k such that for some element r in R, $rx = kx$ for each x in R. The mode of R is a divisor of the characteristic of R. R has mode 1 if and only if R has an identity, and if R has mode 0, then R^* is the unique unital extension of R [4].

If $[(0) : R]_R = (0)$ – that is, if R admits no total zero divisor (literally, if R admits no total zero divisor other than 0) – then Proposition 2.2 implies that there is an element $\alpha = r + ke$ in R^*, where k is the mode of R, such that $\{(n\alpha)\}_{n=0}^{\infty}$ is the set of ideals of R^* lying over (0) in R, and hence $\{R^*/(n\alpha)\}_{n=0}^{\infty}$ is the set of unital extensions of R. We set $R_n = R^*/(n\alpha)$. If $k = 0$, then $\alpha = 0$, and each R_n is R^*; if $k > 0$, then it is easy to prove that $R_n/R \simeq \mathbf{Z}/(nk)$ for each n. It follows that to within R-isomorphism, the rings R_n are distinct (in the terminology of Brown and

McCoy, R_n is not *strictly isomorphic* to R_m for $n \neq m$) if R has nonzero mode. On the other hand, we show presently that for each integer $k > 0$, there is a ring S of mode k such that S has no total zero divisor, but $S_n \simeq S_m$ for all $n,m \geqslant 0$. In summary, we have proved the following result.

Theorem 2.3. *Let R^* be the ring obtained from R by the canonical adjunction of an identity of characteristic* 0.

(1) *If $\{C_\beta\}$ is the set of ideals of R^* lying over* (0) *in R, then $\{R^*/C_\beta\}$ is the set of unital extensions of R. Each C_β is principal as an ideal of R^* and is cyclic as an abelian group. If some C_β is nonzero, then there is a positive integer k, the mode of R, such that each C_β is of the form $(r_\beta + kn_\beta e)$ for some r_β in R and some n_β in* $\mathbf{Z}$.

(2) *If R has no total zero divisor, then there is an element $\alpha = r + ke$ in R^* such that $\{(n\alpha)\}_{n=0}^{\infty}$ is the set of ideals of R^* that lie over* (0) *in R. If $k = 0$, then R^* is the unique unital extension of R. If $k > 0$, then $\{R_n\}_{n=0}^{\infty}$ is the set of unital extensions of R, where $R_n = R^*/(n\alpha)$; the rings R_n are distinct to within R-isomorphism.*

Remark 2.4. If R has characteristic $q > 0$, then $mqe \in [(0) : R]_{R^*}$ and $(mqe) \cap R = (0)$ for each positive integer m. Thus, in the notation of Theorem 2.3, $\{(mqe)\}_{m=1}^{\infty} \subseteq \{C_\beta\}$. This containment may be proper; for example, if R is the ring $4\mathbf{Z}/24\mathbf{Z}$, then the mode of R is 2 and the characteristic of R is 6. The ring $R^*/(mqe)$ is *the ring obtained from R by the canonical adjunction of an identity of characteristic mq* [8, p. 5].

If R has no total zero divisors, then $R_1 = R^*/(\alpha)$ appears, in many ways, to be the most "efficient" unital extension of R. For example, $|R_1/R|$ is minimal in the set $\{|S/R|\}$, as S ranges over all unital extensions of R; if R has a regular element, then the total quotient ring T of R has an identity e' and R_1 is isomorphic to the subring $R[e']$ of T generated by R and e'. To prove this, let $\phi : R^* \to R[e']$ be defined by

$$\phi(r + me) = r + me' .$$

Since R has no total zero divisors (because R has a regular element), there is an element $\alpha = x + ke$ in R^* such that the kernel of ϕ is contained in (α). For each element r in R,

$$0 = \alpha r = xr + kr .$$

In particular, if r is regular in R, then

$$0 = (xr + kr)r^{-1} = x + ke' = \phi(\alpha) ;$$

that is, α is in the kernel of ϕ. Therefore (α) is the kernel of ϕ and $R[e'] \simeq R^*/(\alpha) = R_1$.

We turn to the problem of showing that the rings R_i need not be isomorphically distinct.

Proposition 2.5. *Assume that R is a ring with identity, S is a ring with no total zero divisors, and $T = R \oplus S$. Then T has no total zero divisors, and for each $n \geqslant 0$, $T_n \simeq R \oplus S_n$.*

Proof. It is clear that T has no total zero divisors. We prove first that $T_0 \simeq R \oplus S_0$; that is, $T^* \simeq R \oplus S^*$. If e denotes the identity element of T^*, e' is the identity of R, and e'' is the identity of S^*, then it is straightforward to verify that the mapping

$$(r, s) + ke \leftrightarrow (r + ke', s + ke'')$$

is an isomorphism between the rings T^* and $R \oplus S^*$. If the ring S has mode k, and s is the unique element of S such that $sx = kx$ for each element x in S, then T has mode k and $(ke', s) = \alpha'$ is the unique element of T such that $\alpha' t = kt$ for each t in T. It follows that $T_n = T^*/(n\alpha)$, where $\alpha = (-ke', s) + ke$, for each integer $n \geqslant 0$. Under the isomorphism between T^* and $R \oplus S^*$ described above, α is mapped to $(0, -s + ke'') = \alpha''$ and $n\alpha$ is mapped to $n\alpha''$. Consequently,

$$T_n = T^*/(n\alpha) \simeq (R \oplus S^*)/(n\alpha'') \simeq R \oplus (S^*/(n(-s + ke''))).$$

Finally, since S has mode k and $sx = kx$ for each x in S,

$$S^*/(n(-s + ke'')) = S_n$$

and $T_n \simeq R \oplus S_n$, as asserted in the proposition. □

Example 2.6. Let k be a positive integer, and let S be a ring of mode k without total zero divisors (kZ is such a ring). For each $i,j \geqslant 0$, let R_{ij} be the ring S_i and let R be the complete direct sum of the family $\{R_{ij}\}_{i,j \geqslant 0}$ of rings; R is a ring with identity. If $T = R \oplus S$, then T is a ring of mode k, R has no total zero divisors, and for each $n \geqslant 0$,

$$T_n \simeq R \oplus S_n \simeq R.$$

Thus the family $\{T_i\}_{i=0}^{\infty}$ represents only one isomorphism class, although, to within T-isomorphism, the rings T_i and T_j are distinct for $i \neq j$.

It is clear that by a variation of the construction used in Example 2.6, and by proper choice of the rings S, numerous possibilities for the isomorphism classes of the family $\{T_i\}_{i=0}^{\infty}$, where T is of mode k and has no total zero divisors, can be realized.

3. Dimension theory for ideals

Our purpose in this section is to relate the (Krull) dimension of a ring R to that of an ideal A of R; we restrict to the case in which R is finite-dimensional. In Section 4 we shall consider the case where R is a unital extension of A. Parts of our first result, Proposition 3.1, already appear in [7].

Proposition 3.1. *Assume that A is a proper ideal of the ring R, P is a proper prime ideal of A, and the ideal Q of A is P-primary.*

(1) *Q is an ideal of R and the ideal $[Q : A]_R$ of R lies over Q in A.*

(2) *$\mathcal{P} = [P : A]_R$ is the unique prime ideal of R lying over P in A and $R_{\mathcal{P}} \simeq A_P$.*

(3) *If there is a fixed positive integer n such that $x^n \in Q$ for each x in P, then $[Q : A]_R$ is $[P : A]_R$-primary and $[Q : A]_R$ is the unique primary ideal of R lying over Q in A.*

(4) *If R has an identity and A is maximal in R, then the following conditions are equivalent*:

(a) *P is prime in R.*

(b) *A/P is a ring without identity.*

(c) *Q is primary in R.*

(d) *A/Q is a ring without identity.*

In particular, if R is quasi-local with maximal ideal A, then each prime ideal of A is prime in R.

Proof. Except for the assertion that $R_{\mathcal{P}}$ and A_P are isomorphic, (1)–(3) appear as [7, Proposition 3.1]. We use the following lemma to prove that $R_{\mathcal{P}}$ and A_P are isomorphic; its proof is straightforward and will be omitted (cf. [8, (2.8)]).

Lemma 3.2. *Assume that A is an ideal of the ring R and N is a multiplicative system in R such that $A \cap N \neq \emptyset$. Then R_N is isomorphic to $A_{A \cap N}$.*

To establish (4) of the proposition, we first note that the equivalence of (b) and (d) is proved in [6, Lemma 3]. If (d) fails, then A/Q is a nontrivial direct summand of the ring R/Q, and hence the zero ideal of R/Q is not primary. Consequently, (c) implies (d).

(a) $\Rightarrow$ (c). We have $Q \subseteq P$, and P is contained in the radical of Q in R. Hence to prove that Q is primary in R, we need only prove that if $x, y \in R$ are such that $xy \in Q$ and $x \notin P$, then $y \in Q$. Since P is prime in R, $y \in P$. If $x \in A$, then $y \in Q$ since Q is P-primary as an ideal of A. If $x \notin A$, then $1 = a + bx$ for some a in A and b in R, and $y - ay = bxy \in Q$. Choose z in $A - P$. Since $1 - a \notin P$ and P is prime in R,

$$z(1 - a) = z - az \notin P.$$

But $y(z - az) = (y - ay)z$ is in Q, a primary ideal of A. Consequently, $y \in Q$ and Q is primary in R.

(b) $\Rightarrow$ (d). Consider elements x and y of R such that $xy \in P$. If x and y are in A, then x or y is in P. If, say, $x \notin A$, then $y \in A$, and as in the proof that (a) implies (c), there is an element a in A such that $y - ya \in P$. Since A/P is a ring without identity, there is an element b in A such that $b - ab \notin P$. Consequently, $y \in P$ for $b(y - ay) = y(b - ab) \in P$.

This completes the proof that conditions (4)(a)–(d) are equivalent. □

If R is quasi-local with maximal ideal A, then for each proper prime ideal P of A, R/P is quasi-local with maximal ideal A/P. Thus the ring R/P is indecomposable, and consequently A/P is a ring without identity.

Corollary 3.3. *If M is the maximal ideal of the quasi-local ring R, then* $\dim M = \dim R - 1$.

In Section 5 we shall prove a generalization of Corollary 3.3 (Proposition 5.8) that yields a formula for the dimension of an architrary maximal prime ideal of a ring.

Corollary 3.4. *Assume that A is an ideal of the ring R.*

(1) $\dim A \leqslant \dim R \leqslant \dim A + \dim(R/A) + 1$.

(2) *If there is a maximal prime ideal of R of rank* $\dim R$ *that does not contain A, then* $\dim A = \dim R$; *if no such maximal prime ideal exists, then* $\dim A < \dim R$.

Proof. For (1), see [7, Corollary 3.2].

If $P_0 \subset P_1 \subset \ldots \subset P_k$ is a chain of proper prime ideals of R of length $k = \dim R$, where $A \nsubseteq P_k$, then Proposition 3.1(2) implies that

$$P_0 \cap A \subset P_1 \cap A \subset \ldots \subset P_k \cap A$$

is a chain of distinct proper prime ideals of A. Therefore $\dim A \geqslant k$, and equality holds:

$$\dim A = k = \dim R .$$

If each maximal prime of R of length $\dim R$ contains A, then we take a chain

$$P_0 \subset P_1 \subset \ldots \subset P_t$$

of proper prime ideals of A of length $t = \dim A$. Consequently,

$$[P_0 : A]_R \subset [P_1 : A]_R \subset \ldots \subset [P_t : A]_R$$

is a chain of t distinct primes of R; since $A \nsubseteq [P_t : A]_R$, the hypothesis on R implies that there is a maximal prime P of R such that $[P_t : A]_R \subset P$. Therefore

$$\dim R \geqslant t + 1 > \dim A . \square$$

Corollary 3.5. *Assume that A is an ideal of the ring R such that $\sqrt{A} = R$. Then* $\dim A = \dim R$. *Hence if B is an ideal of R, then* $\dim B = \dim \sqrt{B}$.

Proof. The first statement of the corollary follows from Corollary 3.4 and the fact that no proper radical ideal of R, and hence no proper prime ideal of R, contains A.

The second assertion follows from the first by considering B as an ideal of the ring $\sqrt{B}$.

More generally, under the hypothesis of the corollary, if P is a prime ideal of A, then $\sqrt{_R P}$, the radical of P in R, is the unique prime ideal of R lying over P in A. □

4. Dimension theory of unital extensions

This section represents a specialization of the setting of the previous section to a consideration of the relationship between the dimension of a commutative ring R and the dimension of a unital extension of R. Our first result is a direct corollary to Corollary 3.4.

Proposition 4.1. *If S is a unital extension of R, then*

$$\dim R \leqslant \dim S \leqslant \dim R + 2 ;$$

if S/R is not isomorphic to $\mathbf{Z}$, then

$$\dim S \leqslant \dim R + 1 .$$

It is easy to see that each of the values 0, 1, 2 can be realized as $\dim R^* - \dim R$ for an integral domain R of dimension $n \geqslant 0$ and for the canonical unital extension R^* of R. For example,

$$\dim R^* - \dim R = 0$$

if $R = \{X_1, ..., X_n\} Q[X_1, ..., X_n]$;

$$\dim R^* - \dim R = 1$$

if R is the maximal ideal of a rank $n + 1$ valuation ring of the form $K + R$, where K is a field of nonzero characteristic; and

$$\dim R^* - \dim R = 2$$

if R is the maximal ideal of a rank $n + 1$ valuation ring of the form $K + R$, where K is a field of characteristic 0. Nevertheless, the case in which $\dim R^* - \dim R = 2$ is exceptional, as is shown by the next result.

Proposition 4.2. *Let S be a unital extension of R, where* $\dim R = k \geqslant 0$.

(1) *If* $\dim S = k + 2$, *then R is a prime ideal of S, each chain of proper prime ideals of S of length $k + 2$ contains R, and S is isomorphic to R^*.*

(2) *If R has positive mode, then* $\dim S \leqslant k + 1$.

Proof. If $\dim S = k + 2$, then Proposition 4.1 implies that R is prime in S. If $P_0 \subset P_1 \subset ... \subset P_{k+2}$ is a chain of proper prime ideals of S, then P_{k+1} contains R, for if not,

$$P_0 \cap R \subset P_1 \cap R \subset \ldots \subset P_{k+1} \cap R$$

is a chain of proper prime ideals of R, contrary to the assumption that R has dimension k. Hence $R \subseteq P_{k+1} \subset P_{k+2}$, and because S/R is a homomorphic image of $\mathbf{Z}$, it follows that $R = P_{k+1}$, S/R is isomorphic to $\mathbf{Z}$, and S is isomorphic to R^*, the ring obtained from R by the canonical adjunction of an identity of characteristic 0.

We prove the contrapositive of (2): if $\dim S = k + 2$, then R has mode 0. The proof of (1) shows that there is a prime ideal P of $S \simeq R^*$ properly contained in R. Hence

$$[(0) : R]_S \subseteq [P : R]_S = P \subset R ,$$

and R has mode zero, as asserted. □

We note that in considering the dimension of S and R, where S is a unital extension of R, there is no loss of generality in assuming that R has no total zero divisors. For if $A = \sqrt{[(0) : R]_R}$, then A is contained in each prime ideal of R and in each prime ideal of S so that

$$\dim R = \dim R/A, \qquad \dim S = \dim S/A ,$$

and S/A is a unital extension of R/A; moreover, $[A : R]_R = A$, for if $xr \in A$ for each r in R, then $x^2 \in A$, and since A is a radical ideal, $x \in A$. It follows that R/A has no total zero divisors.

If R is a ring with no total zero divisors and R^* is the ring obtained from R by the canonical adjunction of an identity of characteristic 0, then we recall from Section 2 that there is an element $\alpha = x + ke$ of R^*, where k is the mode of R, such that $(\alpha) = [(0) : R]_{R^*}$ and such that $\{R^*/(n\alpha)\}_{n=0}^{\infty}$ is the set of unital extensions of R. We use this fact presently.

Proposition 4.3. *Assume that the ring R has no total zero divisors and S is a unital extension of R. If* $\dim R^* \geqslant 2$, *then* $\dim S = \dim R^*$.

Proof. Let $P_0 \subset P_1 \subset \ldots \subset P_m$ be a chain of proper prime ideals of R^*, where $\dim R^* = m \geqslant 2$. Since the dimension of $R^*/R \simeq \mathbf{Z}$ is 1, $P_0 \not\supseteq R$ and Proposition 3.1 implies that $P_0 = [P_0 \cap R : R]_{R^*}$. In particular, P_0 contains $[(0) : R]_{R^*} = (\alpha)$. It then follows that

$$P_0/(n\alpha) \subset P_1/(n\alpha) \subset \ldots \subset P_m/(n\alpha)$$

is a chain of proper primes of $R^*/(n\alpha)$ for each $n \geqslant 0$. Consequently,

$$m \leqslant \dim R^*/(n\alpha) \leqslant \dim R^* = m ,$$

and each unital extension of R has dimension m. □

The next two results use the notation of the paragraph preceding the statement of Proposition 4.3. In particular, the ring R is assumed to have no total zero divisors.

Proposition 4.4. *Assume that S is a unital extension of R. If* $\dim R \geq 1$, *then* $\dim S = \dim R^*$. *If* $\dim R = -1$, *then* $\dim R^* = 1$, *and if $S \not\simeq R^*$, then* $\dim S = 0$.

Proof. Assume that $\dim R \geq 1$. If $\dim R^* \geq 2$, then Proposition 4.3 implies that S and R^* have the same dimension. If $\dim R^* = 1$, let $Q_0 \subset Q_1$ be a chain of proper prime ideals of R. Then

$$[Q_0 : R]_{R^*} \subset [Q_1 : R]_{R^*}$$

is a chain of proper primes of R^* and $(\alpha) = [(0) : R]_{R^*}$ is contained in $[Q_0 \cdot R]_{R^*}$. Consequently, $\dim S \geq 1$ and equality holds:

$$\dim S = 1 .$$

Since R has dimension -1 if and only if each element of R is nilpotent, the assertions of the proposition regarding this case are clear. □

Proposition 4.5. *If R is a zero-dimensional ring, then* $\dim R^* \geq 1$. *If* $\dim R^* = 1$, *then either all other unital extensions have dimension* 1 *or they all have dimension* 0.

Proof. It is clear that the dimension of R^* is at least 1. Suppose that $\dim S = 1$ for some unital extension S of R such that $S \not\simeq R^*$. Assume that $(n\alpha)$ is the kernel of the R-homomorphism of R^* onto S, and let

$$(n\alpha) \subseteq Q_0 \subset Q_1 ,$$

where Q_0 and Q_1 are proper prime ideals of R^*. If $Q_0 \cap R \subset R$, then

$$Q_0 = [Q_0 \cap R : R]_{R^*}$$

and Q_0 contains (α); thus each unital extension of R distinct from R^* has dimension 1. If $R \subseteq Q_0$, then, in fact, $Q_0 = R$. But then $(n\alpha) \subseteq R$, and

$$(0) = (n\alpha) \cap R = (n\alpha) .$$

It follows that either $n = 0$ or $\alpha = 0$; that is, either $S \simeq R^*$ or R^* is the unique unital extension of R. □

Propositions 4.2–4.5 contain a complete analysis of the possibilities for $\dim R$ and $\dim S$, where S is a unital extension of R. We summarize these results in Theorem 4.6.

Theorem 4.6. *Assume that R^* is the ring obtained from R by the canonical adjunction of an identity of characteristic* 0, *and let S denote a unital extension of R not isomorphic to R^*. The possibilities for* $\dim R$, $\dim R^*$ *and* $\dim S$ *are given by the following chart*:

dim R	dim R^*	dim S
-1	1	0
0	1	*1 for all S or 0 for all S*
	2	*no such S*
$n \geq 1$	n	n
	$n+1$	$n+1$
	$n+2$	*no such S*

In relation to the chart of Theorem 4.6, we seek next to answer the following two questions.

(1) *What property distinguishes the zero-dimensional rings R such that each unital extension of R has dimension* 1?

(2) *What property distinguishes the n-dimensional rings R* ($n \geq 1$) *such that* $\dim R^* = n$?

Answers to (1) and (2) are contained in Propositions 4.7 and 4.8.

Proposition 4.7. *Assume that R is a zero-dimensional ring such that* $\dim R^* = 1$, *and let $\{P_\lambda\}$ be the set of proper prime ideals of R.*

(a) *If R/P_λ has an identity for each λ, then each unital extension of R that is not isomorphic to R^* has dimension* 0.

(b) *If R/P_λ is a ring without identity for some λ, then each unital extension of R that is not isomorphic to R^* has dimension* 1.

Proof. There is no loss of generality in assuming that R has no total zero divisors and that R has positive mode k. Then in the notation of Section 2, $\{R_i\}_{i=0}^{\infty}$ is the set of unital extensions of R, where $R_i = R^*/(i\alpha)$. To prove the proposition, we need only show that R_i, for $i > 0$, has dimension 0 if each R/P_λ has an identity, or dimension 1 if some R/P_λ has no identity. Since R_i is a homomorphic image of R^*,

$$\dim R_i \leq 1 .$$

For a fixed λ, we let P_λ^* be the unique prime ideal of R_i lying over P_λ in R. The domain R_i/P_λ^* is a unital extension of the domain R/P_λ. Hence if R/P_λ has an identity, then $R_i/P_\lambda^* \simeq R/P_\lambda$, and P_λ^* is maximal in R_i. If R/P_λ has no identity, then R/P_λ is a nonzero proper ideal of R_i/P_λ^*, R_i/P_λ^* is not a field, and P_λ^* is not maximal in R_i. It follows that R_i has dimension 1 if some R/P_λ is a ring without identity. And if each R/P_λ has an identity, then the observations just made show that each chain

$$Q_0 \subset Q_1 \subset \ldots \subset Q_t$$

of proper primes of R_i with $R \nsubseteq Q_0$ has length 0, whereas each chain with $R \subseteq Q_0$ also has length 0 because $R_i/R \simeq \mathbf{Z}/(ki)$, a zero-dimensional ring. This completes the proof of the proposition. □

Proposition 4.8. *Assume that* $\dim R = k \geqslant 1$. *In order that the dimension of* R^* *be equal to the dimension of* R, *the following conditions are necessary and sufficient*:

(1) R/P *has an identity for each prime* P *of* R *of rank* k.

(2) *For each prime* P *of* R *of rank* $k-1$, R/P *has nonzero mode; that is,* P *is not prime in* R^*.

Proof. Suppose that (1) fails and that P is a prime ideal of R of rank k such that R/P does not have an identity. Then as in the proof of Proposition 4.7, $P^* = [P : R]_{R^*}$ is not a maximal ideal of R^*, and hence $\dim R^* > \dim R$. Assume that (2) fails and that P is a prime of R of rank $k-1$ such that R/P has mode 0. Then P is prime in R^*, and if

$$P_0 \subset P_1 \subset \dots \subset P_{k-1} = P$$

is a chain of prime ideals of R of length $k-1$, then

$$[P_0 : R]_{R^*} \subset \dots \subset [P_{k-2} : R]_{R^*} \subset P \subset R \subset R + 2R^*$$

is a chain of primes of R^* of length $k+1 > \dim R$. Thus, if $\dim R^* = k$, then (1) and (2) are satisfied.

Next assume that there is a unital extension S of R such that $\dim S > \dim R$, and let

$$Q_0 \subset Q_1 \subset \dots \subset Q_t$$

be a chain of prime ideals of S of length $t = \dim S$. It follows that Q_t contains R. If $Q_t = R$, then $Q_0 \subset \dots \subset Q_{t-1}$ is a chain of proper prime ideals of R, so $\dim R \geqslant t-1$. But Q_{t-1} has rank $t-1$ and the mode of R/Q_{t-1} is zero, so (1) fails. Thus assume that $Q_t \supset R$. If Q_{t-1} contains R, then $Q_{t-1} = R$ and $S = R^*$. Then $Q_0 \subset \dots \subset Q_{t-2}$ is a chain of prime ideals R and the rank of Q_{t-2} is at least $k-1$, although R/Q_{t-2} has mode zero. If Q_{t-1} does not contain R, then

$$Q_0 \cap R \subset \dots \subset Q_{t-1} \cap R$$

is a chain of prime ideals of R and $k = t-1$. But then $Q_{t-1} \cap R$ has rank k, while $R/(Q_{t-1} \cap R)$ does not have an identity, for this would imply that Q_{t-1} is maximal in S. □

5. Dimension sequences

For the ring R, we let $R^{(m)}$ denote the polynomial ring $R[X_1, \dots, X_m]$ in m indeterminates over R. If R has finite dimension n_0, then $\dim R^{(m)} = n_m$ is finite for each positive integer m, and the sequence $\{n_i\}_{i=0}^{\infty}$ is called the *dimension sequence* for R. In [1], Arnold and Gilmer have determined all sequences of nonnegative integers that can be realized as the dimension sequence for a ring with identity. In this section we consider the same problem for rings without identity, and in Theorem 5.10

we show that for rings with nonnegative dimension, the set of dimension sequences for rings without identity coincides with the set of dimension sequences for rings with identity. We begin by considering the trivial case $\dim R = -1$.

Proposition 5.1. *If* $\dim R = -1$, *then* $\dim R^{(n)} = -1$ *for each nonnegative integer* n.

Proof. The dimension of R is -1 if and only if each element of R is nilpotent. But if this occurs, then it is also the case that each element of $R^{(n)}$ is nilpotent. Therefore $\dim R^{(n)} = -1$. □

Throughout the remainder of this section R denotes a ring with nonnegative finite dimension. In order to determine the set of possible dimension sequences for such a ring, we must first extend to rings without identity certain results that are known for rings with identity.

Proposition 5.2. *If* A *is a radical ideal of* $R^{(n)}$ *and* $B = A \cap R$, *then* $B^{(n)} \subseteq A$.

Proof. It suffices to show that each element of the form bm is in A, where $b \in B$ and $m = X_1^{e_1} \ldots X_n^{e_n}$ for some nonnegative integers $e_1, \ldots, e_n$. Since $b \in B \subseteq A$ and $bm^2 \in R^{(n)}$, it follows that $b(bm^2) = (bm)^2 \in A$. Since A is a radical ideal, $bm \in A$ as we wished to show. □

If P is a proper prime ideal of $R^{(n)}$, then it is an immediate consequence of Proposition 5.2 that $P \cap R$ is a proper prime ideal of R, for if $P \cap R = R$, then $R^{(n)} \subseteq P$, contrary to the assumption that P is proper. Throughout the remainder of this section we use $r(P)$ to denote the rank of the proper prime ideal P of the ring R; that is, $r(P) = \dim R_P$.

Proposition 5.3. *Let* $\{M_\lambda\}$ *be the set of maximal prime ideals of* R.

(1) $\dim R^{(n)}_{M_\lambda} \leq \dim R^{(n)}$ *for each* λ.

(2) *If* $P_0 \subset P_1 \ldots \subset P_t$ *is a chain of proper prime ideals of* $R^{(n)}$ *of length* $t = \dim R^{(n)}$ *then* $M = P_t \cap R$ *is a maximal prime ideal of* R *and* $\dim R^{(n)}_M = \dim R^{(n)}$.

Proof. For a fixed λ, if we set $N_\lambda = R - M_\lambda$, then

$$R^{(n)}_{M_\lambda} = R^{(n)}_{N_\lambda} \cong (R^{(n)})_{N_\lambda}.$$

Hence $\dim R^{(n)}_{M_\lambda} \leq \dim R^{(n)}$ for each λ. Now let

$$\mathcal{C}: \qquad P_0 \subset P_1 \subset \ldots \subset P_t$$

be a chain of proper prime ideals of $R^{(n)}$ of length $t = \dim R^{(n)}$. It follows from Proposition 5.2 that $P = P_t \cap R$ is a proper prime ideal of R. Since R is assumed to have finite dimension, there exists a maximal prime ideal M of R such that $P \subseteq M$.

The chain $\mathcal{C}$ extends to a chain of prime ideals of $R_M^{(n)}$ of length t, so we have

$$t \leqslant \dim R_M^{(n)} \leqslant \dim R^{(n)} = t .$$

Therefore $t = \dim R_M^{(n)}$. The ring R_M has an identity, and if P_t^e denotes the extension of P_t to $R_M^{(n)}$, then

$$r(P_t^e) = t = \dim R_M^{(n)} .$$

By [1, Corollary 2.9], $P_t^e \cap R_M$ is a maximal ideal of R_M. It follows that $P = M$. □

Corollary 5.4. *If R is Noetherian with $\dim R = k$, then $\dim R^{(n)} = k + n$ for each positive integer n.*

Proof. Let $\{M_\lambda\}_{\lambda \in \Lambda}$ be the set of maximal prime ideals of R. For each λ in Λ, R_{M_λ} is a Noetherian ring with identity, so

$$\dim R_{M_\lambda}^{(n)} = \dim R_{M_\lambda} + n .$$

It follows from Proposition 5.3 that

$$\dim R^{(n)} = \sup_\lambda \{\dim R_{M_\lambda}^{(n)}\} = n + \sup_\lambda \{\dim R_{M_\lambda}\} = n + \dim R = n + k. \ \square$$

Corollary 5.5. *If $\dim R = 0$, then $\dim R^{(n)} = n$ for each positive integer n.*

Proof. For each maximal ideal M of R, R_M is a zero-dimensional ring with identity. Hence $\dim R_M^{(n)} = n$ for each positive integer n. The corollary then follows from Proposition 5.3. □

A natural approach to characterizing the dimension sequences for rings without identity would be to obtain a "nice" relationship between the dimension sequence of a ring and that of a unital extension. That one cannot, in general, expect to obtain such a relationship is illustrated by Example 5.14. We show, however, in the following result that for a certain class of rings, the dimension sequence for the ring coincides with that of any unital extension.

Proposition 5.6. *Let R be a ring such that $\dim R \geqslant 1$ and such that R/P has an identity for each proper prime ideal P of R. If S is a unital extension of R, then for each nonnegative integer n,*

$$\dim R^{(n)} = \dim S^{(n)} .$$

Proof. For $n = 0$, the result follows from Propositions 4.7 and 4.8. Thus assume that $n > 0$, and let Q be a prime ideal of $S^{(n)}$ such that $Q \supseteq R^{(n)}$. We show that there is no prime ideal P of $S^{(n)}$ such that $Q \supseteq P$ and $P \not\supseteq R^{(n)}$. Otherwise, $Q_0 = Q \cap S$ is a prime ideal of S such that

$$Q_0 \supseteq R, \qquad Q_0 \supseteq P_0 = P \cap S, \qquad P_0 \not\supseteq R.$$

If we set $P_0' = P_0 \cap R$, then R/P_0' has an identity; that is, there exists $t \in R - P_0'$ such that $tr - r \in P_0'$ for each $r \in R$. Therefore, if e is the identity for S, then $-t + e$ is in $[P_0' : R]_S = P_0 \subseteq Q_0$. Since $-t \in R \subseteq Q_0$, it follows that $Q_0 = S$, and hence that $Q = S^{(n)}$. Now suppose that $\dim S^{(n)} = k$ and let

$$Q_0 \subset \ldots \subset Q_k$$

be a chain of prime ideals of $S^{(n)}$ of length k. If $Q_k \supseteq R^{(n)}$, then we have just shown that $Q_0 \supseteq R^{(n)}$ also. Therefore

$$Q_0/R^{(n)} \subset \ldots \subset Q_k/R^{(n)}$$

is a chain of prime ideals of $S^{(n)}/R^{(n)} \simeq (S/R)^{(n)}$, and since S/R is a homomorphic image of $\mathbf{Z}$, it follows that $k \leqslant n+1$. Since $\dim R \geqslant 1$, it is an easy consequence of Proposition 5.3 that $n + 1 \leqslant \dim R^{(n)}$. Since $R^{(n)}$ is an ideal of $S^{(n)}$,

$$\dim R^{(n)} \leqslant \dim S^{(n)},$$

so it follows that $\dim R^{(n)} = \dim S^{(n)}$. If $Q_k \not\supseteq R^{(n)}$, then

$$Q_0 \cap R^{(n)} \subset \ldots \subset Q_k \cap R^{(n)}$$

is a chain of $k+1$ distinct proper prime ideals of $R^{(n)}$. Therefore we have that

$$k \leqslant \dim R^{(n)} \leqslant \dim S^{(n)} = k,$$

so $\dim R^{(n)} = \dim S^{(n)}$. □

We mention an alternate proof of Proposition 5.6 that is based on Proposition 5.3. Thus, with the hypothesis of Proposition 5.6, let M_α be a maximal prime ideal of S. If M_α does not contain R, then $S_{M_\alpha} \simeq R_{M_\alpha \cap R}$. If M_α contains R, then the proof of Proposition 5.6 shows that each prime of S contained in M_α contains R; hence M_α has height 1, and R is the unique prime of S properly contained in M_α if $S/R \simeq \mathbf{Z}$, while M_α has height 0 otherwise. It follows that either S_{M_α} is zero-dimensional or S_{M_α} is one dimensional with a unique minimal prime ideal, the extension of R to S_{M_α}; moreover, since $S/R \simeq \mathbf{Z}$, S_{M_α} has dimension sequence 1, 2, 3, ... in the latter case. Since $\dim R \geqslant 1$, it follows that $\max\{S_{M_\alpha}^{(n)}\}$ is the same as $\max\{S_{M_\alpha}^{(n)} \mid R \not\subseteq M_\alpha\}$, and because $S_{M_\alpha} \simeq R_{M_\alpha \cap R}$,

$$\dim S^{(n)} \leqslant \dim R^{(n)} \leqslant \dim S^{(n)}$$

for each n. The proof of our next result is based on considerations similar to those we have just made.

Theorem 5.7. *Each dimension sequence for a ring without identity is also the dimension sequence for a ring with identity.*

Proof. Let s be the dimension sequence for the ring R. If $\dim R = 0$, then it follows from Corollary 5.5 that s is the dimension sequence for each ring (with or without identity) of dimension zero. Thus we assume that $\dim R \geqslant 1$. It follows from Proposition 5.3 that

$$s = \{\max_{\lambda} \{\dim R_{M_\lambda}^{(n)}\}\}_{n=0}^{\infty},$$

where $\{M_\lambda\}_{\lambda \in \Lambda}$ is the set of maximal prime ideals of R. If we set $S = \Sigma_{\lambda \in \Lambda} R_{M_\lambda}$ and if

$$Q_{\lambda_0} = M_{\lambda_0}^e \oplus \sum_{\lambda \neq \lambda_0} R_{M_\lambda},$$

where $M_{\lambda_0}^e$ is the extension of M_{λ_0} to $R_{M_{\lambda_0}}$, then $\{Q_\lambda\}_{\lambda \in \Lambda}$ is the set of maximal ideals of S. Moreover, $S_{Q_\lambda} = R_{M_\lambda}$ for each λ in Λ. It is now an easy consequence of Proposition 5.3 that S also has dimension sequence s. Since each R_{M_λ} has an identity, it follows that S/P has an identity for each proper prime ideal P of S. If T is any unital extension of S, then by Proposition 5.6, T has dimension sequence s. □

We obtain the converse of Theorem 5.7 by considering the dimension sequence for an ideal A of a ring R. Our next result gives a formulation of the dimension sequence for A in terms of prime ideals of R.

Proposition 5.8. *Let A be an ideal of the ring R, and let $\{P_\lambda\}_{\lambda \in \Lambda}$ be the set of prime ideals of R that are maximal with respect to not containing A. If A is considered as a ring, then*

$$\dim A = \max_{\lambda \in \Lambda} \{r(P_\lambda)\}$$

and the dimension sequence for A is

$$\{\max_{\lambda \in \Lambda} \{\dim R_{P_\lambda}^{(n)}\}\}_{n=0}^{\infty}.$$

Proof. The first assertion is equivalent to the second for the case $n = 0$, and hence we prove only the second assertion. If $k = \dim R_{P_\lambda}^{(n)}$, then there exists a chain

$$Q_0 \subset \ldots \subset Q_k$$

of $k + 1$ prime ideals of $R^{(n)}$ such that

$$Q_k \cap A^{(n)} \subset \ldots \subset Q_k \cap A^{(n)}$$

is a chain of $k+1$ distinct proper prime ideals of $A^{(n)}$. It follows that

$$\dim A^{(n)} \geqslant \max_{\lambda \in \Lambda} \{\dim R_{P_\lambda}^{(n)}\}.$$

Suppose that $\dim A^{(n)} = t$ and let

$$P_0 \subset \ldots \subset P_t$$

be a chain of $t+1$ proper prime ideals of $A^{(n)}$. If

$$P_i' = [P_i : A^{(n)}]_{R^{(n)}}$$

for $0 \leqslant i \leqslant t$, then $P_0' \subset \ldots \subset P_t'$ is a chain of $t+1$ distinct prime ideals of $R^{(n)}$. Moreover, since $P_t' \not\supseteq A^{(n)}$, there exists $\lambda \in \Lambda$ such that $P_t' \cap R \subseteq P_\lambda$. But then $\dim R_{P_\lambda}^{(n)} \geqslant t$, so the result follows. □

Before proceeding with our next result, Theorem 5.9, we review some notation and terminology introduced in [1].

If $s = \{n_i\}_{i=0}^{\infty}$ is a sequence of nonnegative integers, then the sequence $\{n_i - n_{i-1}\}_{i=1}^{\infty}$ is called the *difference sequence* for s. Let $\mathcal{S}$ be the set of sequences $s = \{n_i\}_{i=0}^{\infty}$ of nonnegative integers such that the associated difference sequence $\{d_i\}_{i=1}^{\infty}$ has the following properties:

(1) $n_0 + 1 \geqslant d_1 \geqslant d_2 \geqslant \ldots$;

(2) there is a positive integer k such that $1 \leqslant d_k = d_{k+1} = \ldots$.

For $s_1, \ldots, s_r$ in $\mathcal{S}$, where $s_i = \{n_j^{(i)}\}_{j=0}^{\infty}$, $\sup\{s_1, \ldots, s_r\}$ is defined to be the sequence $s = \{n_j\}_{j=0}^{\infty}$, where $n_j = \sup\{n_j^{(1)}, \ldots, n_j^{(r)}\}$ for each $j \geqslant 0$. In [1], Arnold and Gilmer have shown that

$$\mathcal{D} = \{\sup\{s_i\}_1^r \mid s_1, \ldots, s_r \in \mathcal{S}\}$$

is the set of sequences of nonnegative integers that can be realized as the dimension sequence for a ring with identity. In fact, if $s \in \mathcal{D}$ with $s = \sup\{s_1, \ldots, s_r\}$, where $s_i \in \mathcal{S}$ for $1 \leqslant i \leqslant r$, then it is shown in [1, Theorem 4.10] that there exists a semi-quasi-local domain D with maximal ideals $M_1, M_2, \ldots, M_r$ such that D_{M_i} has dimension sequence s_i and D has dimension sequence s. This construction will be useful in the proof of our next result.

Theorem 5.9. *Each dimension sequence for a ring with identity is also the dimension sequence for an integral domain without identity.*

Proof. Suppose that $s = \{n_i\}_{i=0}^{\infty}$ is the dimension sequence for a ring with identity. As in Theorem 5.7, we assume that $n_0 \geqslant 1$. Since $s \in \mathcal{D}$, there exist sequences $s_1, \ldots, s_r \in \mathcal{S}$ such that $s = \sup\{s_1, \ldots, s_r\}$. Set $s_0 = \{i+1\}_{i=0}^{\infty}$. Then $s_0 \in \mathcal{S}$, and since $n_0 \geqslant 1$, we have $s = \sup\{s_0, s_1, \ldots, s_r\}$. Let D be a semi-quasi-local domain with dimension sequence s and with maximal ideals $\{M_i\}_{i=0}^{r}$ such that D_{M_i} has dimension sequence s_i for $0 \leqslant i \leqslant m$ [1, Theorem 4.10]. M_0 is an integral domain without identity and we show that M_0 has dimension sequence s. Since $s = \sup\{s_1, ..., s_r\}$, it follows that

$$\dim D^{(n)} = \max_{1 \leqslant i \leqslant r} \{D_{M_i}^{(n)}\}.$$

It is now an immediate consequence of Proposition 5.8 that $\dim M_0^{(n)} = \dim D^{(n)}$ for each nonnegative integer n. □

Combining Theorems 5.7 and 5.9, we obtain the principal result of this section.

Theorem 5.10. *A sequence of nonnegative integers is the dimension sequence for a ring without identity if and only if it is the dimension sequence for a ring with identity.*

Corollary 5.11. *Let $\{n_i\}_{i=0}^{\infty}$ be the dimension sequence for the ring R, and let $\{d_i\}_{i=1}^{\infty}$ be the corresponding difference sequence. Then $1 \leqslant d_i \leqslant n_0 + 1$ for each i and for some positive integer m, the sequence $\{d_i\}_{i=m}^{\infty}$ is constant.*

Proof. Since $\{n_i\}_{i=0}^{\infty}$ is the dimension sequence for a ring with identity, the corollary follows immediately from [1, Theorem 4.1]. □

A question that arises in relation to Proposition 5.8 is the following. If the ring R has a "nice" dimension sequence and A is an ideal of R, is it also the case that A has a "nice" dimension sequence? For example, if R is Noetherian, then $\{n + \dim R\}_{n=0}^{\infty}$ is the dimension sequence for R. If A is an ideal of R, then is $\{n + \dim A\}_{n=0}^{\infty}$ the dimension sequence for A? Following the notation of [1], we denote by $\mathcal{R}_n$ the class of commutative rings R (with or without identity) such that $r(Q) = r(Q^{(n)})$ for each proper prime ideal Q of R. It is a consequence of Proposition 5.3 and [1, Corollary 2.9] that if R is a ring of dimension k and $R \in \mathcal{R}_n$, then $\dim R^{(n)} = k + n$. In particular, if $R \in \mathcal{R}_{\infty}$, where

$$\mathcal{R}_{\infty} = \bigcap_{n=1}^{\infty} \mathcal{R}_n ,$$

then $\{k + n\}_{n=0}^{\infty}$ is the dimension sequence for R. Prüfer domains and Noetherian rings are in $\mathcal{R}_{\infty}$. In the following result we show that if $R \in \mathcal{R}_n$, then so is each nonzero ideal of R.

Proposition 5.12. *If $R \in \mathcal{R}_n$, then each ideal of R is in $\mathcal{R}^{(n)}$. In particular, if $R \in \mathcal{R}_{\infty}$ and A is an ideal of R, then $A \in \mathcal{R}_{\infty}$, so the dimension sequence for A is $\{n + \dim A\}_{n=0}^{\infty}$.*

Proof. Let A be an ideal of R, where $R \in \mathcal{R}_n$, and let P be a proper prime ideal of A. If $P_1 = [P : A]_R$, then Proposition 3.1 (2) implies that $r(P_1) = r(P)$. Further

$$P_1^{(n)} = [P^{(n)} : A^{(n)}]_{R^{(n)}},$$

so $r(P_1^{(n)}) = r(P^{(n)})$. Since $R \in \mathcal{R}_n$, we have

$$r(P^{(n)}) = r(P_1^{(n)}) = r(P_1) = r(P),$$

hence $A \in \mathcal{R}_n$. □

Proposition 5.12 yields an alternate proof of Corollary 5.4, for if R is a Noetherian ring, then R is an ideal of the Noetherian ring R^*.

In Proposition 5.8 we showed that if S is a unital extension of R, then the sequence $\{\dim S^{(n)} - \dim R^{(n)}\}_{n=0}^{\infty}$ may be the constant sequence 0, 0, 0, ... We shall now give bounds on $\dim S^{(n)} - \dim R^{(n)}$ and give an example to show that, in general, the sequence $\{\dim S^{(n)} - \dim R^{(n)}\}$ need not be well-behaved.

Proposition 5.13. *Let S be a unital extension of the ring R. For each nonnegative integer n,*

$$0 \leq \dim S^{(n)} - \dim R^{(n)} \leq n + 2 .$$

Proof. $R^{(n)}$ is an ideal of $S^{(n)}$, so by Corollary 3.4(1),

$$\dim R^{(n)} \leq \dim S^{(n)} ;$$

that is, $0 \leq \dim S^{(n)} - \dim R^{(n)}$. Suppose that $\dim S^{(n)} = k$ and let

$$Q_0 \subset Q_1 \subset \ldots \subset Q_k$$

be a chain of prime ideals of $S^{(n)}$. If t is the smallest integer such that $Q_t \supseteq R^{(n)}$, then

$$Q_t/R^{(n)} \subset \ldots \subset Q_k/R^{(n)}$$

is a chain of prime ideals of $S^{(n)}/R^{(n)} \cong (S/R)^{(n)}$. Since S/R is a homomorphic image of $\mathbf{Z}$, it follows that $k - t + 1 \leq n + 2$. Since

$$R^{(n)} \cap Q_0 \subset \ldots \subset R^{(n)} \cap Q_{t-1}$$

is a chain of distinct prime ideals of $R^{(n)}$,

$$\dim S^{(n)} = k = (t - 1) + (k - t + 1) \leq \dim R^{(n)} + n + 2 .$$

Thus

$$\dim S^{(n)} - \dim R^{(n)} \leq n + 2 ,$$

as we wished to show. □

That the bounds given in Proposition 5.13 are the best that can be attained is illustrated in the following example.

Example 5.14. Let V be a rank 4 valuation ring of the form $V = K + M$, where M is the maximal ideal of V and K is a field containing Q, with tr.d.$(K/Q) = 2$. It follows from Corollary 3.3 and Proposition 5.12 that $\{3+i\}_{i=0}^{\infty}$ is the dimension sequence for M. $M^* = Z + M$ and by [2, Corollary 5.5], the sequence $\{\dim M^{*(n)}\}_{n=0}^{6}$ is $\{5, 7, 9, 10, 11, 12, 13\}$. Now let D be a domain with identity that has dimension sequence $\{2n+1\}_{n=0}^{\infty}$. If $R = M \oplus D$, then $R^* = M^* \oplus D$ by Proposition 2.5. One can now easily derive the following:

	$n=0$	1	2	3	4	5	6
$\dim R^{(n)}$	3	4	5	7	9	11	13
$\dim R^{*(n)}$	5	7	9	10	11	12	13
$\dim R^{*(n)} - \dim R^{(n)}$	2	3	4	3	2	1	0

Thus, for $n = 0$, 1 or 2,

$$\dim R^{*(n)} - \dim R^{(n)} = n + 2\,,$$

while

$$\dim R^{*(n)} - \dim R^{(n)} = 0$$

for $n = 6$.

In seeking an overring S of $R^{(n)}$ such that S has an identity and such that $\dim S - \dim R^{(n)}$ is related in some nice way (equal, for instance) to $\dim R^* - \dim R$, we ask if there exists a unital extension S of $R^{(n)}$ such that

$$\dim S - \dim R^{(n)} = \dim R^* - \dim R\,.$$

The above example shows that this is not the case. For each integer $n \geqslant 0$ we have $R^{(n)} = M^{(n)} \oplus D^{(n)}$, so if S is any unital extension of $R^{(n)}$, Proposition 2.5 implies that $S = T \oplus D^{(n)}$, where T is some unital extension of $M^{(n)}$. Since $\dim T - \dim M^{(n)} \leqslant 2$, it follows that

$$\dim S - \dim R^{(n)} = 9 - 9 = 0$$

for $n = 4$. But we have already seen that $\dim R^* - \dim R = 2$.

6. Chain theorems for prime ideals

If R is a commutative ring with identity, there are some theorems concerning chains of prime ideals in $R^{(n)}$ that are extremely useful in studying the dimension theory of $R^{(n)}$. Many of these theorems can be extended to rings without identity, and while we do not make use of these results in this paper, we feel that their utility in rings with identity warrants their inclusion here. Each of these results can be proved using either of two methods. The first method is to pass to an appropriate localization R_P of the ring R. Since this localization has an identity, the result is known over R_P. A second method is to pass to a unital extension S of R and apply Proposition 3.1(2). Having thus indicated the general method of proof, we shall omit the proof of each result of this section.

Proposition 6.1. *Let P be a proper prime ideal of the ring R. If*

$$\mathcal{C}: \qquad Q_0 \subset \ldots \subset Q_t$$

is a chain of prime ideals of $R^{(m)}$ such that

$$Q_0 \cap R = \ldots = Q_t \cap R = P,$$

then $t \leqslant n$, and equality holds if and only if $\mathcal{C}$ is a maximal chain of prime ideals lying over P.

If $\mathcal{C}$ denotes the chain $Q_0 \subsetneq \ldots \subsetneq Q_k$ of prime ideals of R, then we call Q_k the *terminal element* of the chain $\mathcal{C}$. A chain $\mathcal{C}$ of proper prime ideals of $R^{(m)}$ is called a *special chain* if for each P in $\mathcal{C}$, $(P \cap R)^{(m)}$ is in $\mathcal{C}$. With this terminology, we state the following result.

Proposition 6.2. *The following equivalent statements hold in $R^{(m)}$:*

(1) *The dimension of $R^{(m)}$ can be realized as the length of a special chain of prime ideals of $R^{(m)}$.*

(2) *If P is a prime ideal of $R^{(m)}$ of rank t, then P is the terminal element of a special chain $\mathcal{C}$ of prime ideals of $R^{(m)}$ such that $\mathcal{C}$ has length t.*

(3) *If P is a prime ideal of $R^{(m)}$ and $Q = P \cap R$, then*

$$r(P) = r(Q^{(m)}) + r(P/Q^{(m)}).$$

For rings with identity, (1) is proved as [9, Theorème 3, p. 35] and is called the Special Chain Theorem. In fact, Jaffard's proof of his Theorème 3 establishes statement (2). A proof of (3) is given in [3, Theorem 1], and the equivalence of (1), (2) and (3) is noted both in [3] and in [1].

Let $\mathcal{C}_m$ denote a chain of prime ideals of $R^{(m)}$, $m \geqslant 1$, and for $0 < i < m$ set $\mathcal{C}_i = \mathcal{C}_m \cap R^{(i)}$. We call $\mathcal{C}_m$ a k^*-chain of prime ideals of $R^{(m)}$ provided the following conditions are satisfied, where $\mathcal{L}(\mathcal{C}_i)$ denotes the length of the chain $\mathcal{C}_i$ of prime ideals:

(1) $\mathcal{L}(\mathcal{C}_m) = k$.

(2) $1 \leqslant \mathcal{L}(\mathcal{C}_i) - \mathcal{L}(\mathcal{C}_{i-1}) \leqslant \mathcal{L}(\mathcal{C}_0) + 1$ for $1 \leqslant i \leqslant m$.

(3) $\mathcal{L}(\mathcal{C}_i) - \mathcal{L}(\mathcal{C}_{i-1}) \leqslant \mathcal{L}(\mathcal{C}_{i-1}) - \mathcal{L}(\mathcal{C}_{i-2})$ for $2 \leqslant i \leqslant m$.

For rings with identity, k^*-chains are defined in [1] and are extremely useful in determining the set of dimension sequences for rings with identity. The following result (for rings with identity) is [1, Theorem 3.3].

Proposition 6.3. *Let Q be a maximal prime ideal of $R^{(m)}$ $(m \geqslant 1)$, with $r(Q) = k$. Then Q is the terminal element of a k^*-chain of prime ideals of $R^{(m)}$.*

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